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LETTER TO THE EDITOR

On the polynomial zeros of the Clebsch–Gordan and Racah coefficients

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Abstract. It is shown that the sybmolic binomial expansions for the Clebsch–Gordan and Racah coefficients are exact for $n=1$ (where $n+1$ indicates the number of terms in the series expansions). When exact, these binomial forms reveal polynomial zeros of degree one, which are trivial structure zeros, hitherto considered as 'non-trivial' zeros, along with polynomial zeros of degree $\geqslant 2$.

The only systematic studies (Koozekanani and Biedenharn 1974, Varshalovich *et al* 1975, Bowick 1976, Biedenharn and Louck 1981) of the polynomial or non-trivial zeros of the Clebsch–Gordan and Racah coefficients are numerical in approach. Koozekanani and Biedenharn (1974) tabulated the non-trivial zeros of the Racah coefficient for arguments $\leqslant \frac{37}{2}$, while Varshalovich *et al* (1975) listed the same for the $3n-j$ symbols for $(j_1+j_2+j_3=)J\leqslant 27$. Bowick (1976) reduced these listings by taking into account the Regge symmetries (Regge 1958, 1959) for these coefficients.

In Srinavasa Rao (1978) a set of six series representations for the Clebsch–Gordan $(3-j)$ coefficient was defined, in the following compact notation:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left\| \begin{matrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{matrix} \right\|$$

$$= \|R_{ik}\|$$

$$= \delta(m_1+m_2+m_3) \prod_{i,k=1}^{3} [R_{ik}!/(J+1)]^{1/2} (-1)^{\sigma(pqr)}$$

$$\times \sum_s (-1)^s [s!(R_{2p}-s)!(R_{3q}-s)!$$

$$\times (R_{1r}-s)!(s+R_{3r}-R_{2p})!(s+R_{2r}-R_{3q})!]^{-1}, \qquad (1)$$

for all six permutations of $(pqr)=(123)$ with

$$\sigma(pqr) = \begin{cases} R_{3p}=R_{2q} & \text{for even permutations.} \\ J+R_{3p}-R_{2q} & \text{for odd permutations.} \end{cases}$$

Following the procedure given by Sato (1955), and using the symbolic notation:

$$p^{(x)} = p!/(p-x)!$$

we obtain

$$\|R_{ik}\| = \delta(m_1+m_2+m_3) \prod_{i,k=1}^{3} [R_{ik}!/(J+1)]^{1/2}(-1)^{\sigma(pqr)}$$
$$\times[\Gamma(n+1, C_u+1, C_v+1, B_{rp}+n+1, B_{rq}+n+1)]^{-1}$$
$$\times\{(B_{rp}+n)(B_{rq}+n)-C_uC_v\}^{(n)} \tag{2}$$

where $\Gamma(a, b, \ldots) = \Gamma(a)\Gamma(b)\ldots$, $n = \min(R_{2p}, R_{3q}, R_{1r})$. C_u, C_v represent the two R_{ik}'s in the triple (R_{2p}, R_{3q}, R_{1r}) other than their minimum, $B_{rp} = R_{3r} - R_{2p}$ and $B_{rq} = R_{2r} - R_{3q}$. The expression (2) given here is general when compared with the result obtained by Sato and Kaguei (1972), as their particular result can be derived by putting $(pqr) = (231)$ in (2), while (2) itself holds for all the six permutations of (pqr).

Equation (2) is symbolic for $n \geq 2$, since it uses the generalised power (Smorodinskii and Shelepin 1972) (the analogue of ordinary power) $p^{(x)} = p!/(p-x)!$, but it is *exact* for $n=1$, since $p^{(1)} = p$. Therefore, the binomial form for the Clebsch–Gordan coefficient explicitly reveals further structural or trivial zeros of this coefficient. We find that of the 39 reduced 'non-trivial' zeros listed by Bowick (1976) and Biedenharn and Louck (1981) for $0 \leq J(=j_i+j_2+j_3) \leq 27$, 21 are trivial structure zeros. In fact, the conventional expansion for the Clebsch–Gordan coefficient, given by equation (1), can be multiplied by a simple factor $(1-\delta_{x,y})$ where

$$X = R_{mr}R_{kp} \qquad \text{and} \qquad Y = R_{mp}R_{kr} \tag{3}$$

provided the number of terms is governed by the power of the binomial, $n = \min(R_{2p}, R_{3q}, R_{1r})$, and it being (say) R_{lq}, for cyclic permutations of $(pqr) = (123) = (lmk)$.

This same condition can also be derived from the set of ${}_3F_2$ $(ABC;\ DE;\ 1)$s (Srinivasa Rao 1978) for the 3j coefficient, when the expansion ends after the second term in the series, (i.e. $1 + ABC/DE = 0$, or $ABC = -DE$) so that these zeros from the polynomial part of the 3-j coefficient can be called structure zeros of degree one. This multiplicative factor $(1-\delta_{X,Y})$ will thus represent all those trivial structure zeros of degree one, considered hitherto as 'non-trivial' zeros.

The Racah coefficient (Racah 1942) is defined as:

$$W(abcd;ef) = (-1)^{a+b+c+d}\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$$
$$= N(-1)^{\beta_1}\sum_P (-1)^P(P+1)!\left[\prod_{i=1}^{4}(P-\alpha_i)!\prod_{j=1}^{3}(\beta_j-P)!\right]^{-1}, \tag{4}$$

where the range of P is restricted to non-negative integral values of the factorials:

$$\max(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq P \leq \min(\beta_1, \beta_2, \beta_3),$$

with

$$\alpha_1 = a+b+e, \qquad \beta_1 = a+b+c+d,$$
$$\alpha_2 = c+d+e, \qquad \beta_2 = a+d+e+f,$$
$$\alpha_3 = a+c+f, \qquad \beta_3 = b+c+e+f,$$
$$\alpha_4 = b+d+f,$$
$$N = \Delta(abe)\cdot\Delta(cde)\cdot\Delta(acf)\cdot\Delta(bdf),$$

and

$$\Delta(xyz) = [(x+y-z)!(x-y+z)!(-x+y+z)!/(x+y+z+1)!]^{1/2},$$

which vanishes unless the usual triangular conditions in x, y and z, namely $|x-y| \leq z \leq x+y$, is satisfied. Thus, the factor N contains all the trivial structure zeros of the Racah coefficient.

Sato (1955) has rearranged (4), by putting $\alpha_0 = \alpha_{\max}$, $\beta_0 = \beta_{\min}$, $n = \beta_0 - \alpha_0$, $P - \alpha_0 = z$, $0 \leq z \leq n$, $A_i = \alpha_0 - \alpha_i$, $B_j = \beta_j - \alpha_0$, with the indices $i = p, q, r$ and $j = u, v$ representing those of α's and β's other than α_0 and β_0 and formally regarding $p^{(x)}$ as if they were powers of p, but actually corresponding to the notations:

$$P^{(x)} = P!/(p-x)! \qquad \text{and} \qquad P^{(-x)} = (p+x)!/P!,$$

into the following symbolic binomial form:

$$\begin{aligned} W(abcd; ef) = {} & N(-1)^{\alpha_0+\beta_1}\Gamma(\alpha_0+2) \\ & \times[\Gamma(n+1, A_p+n+1, A_q+n+1, A_r+n+1, B_u+1, B_v+1)]^{-1} \\ & \times\{(A_p+n)(A_q+n)(A_r+n) - B_uB_v(a_0+1)^{(-1)}\}^{(n)}. \end{aligned} \tag{5}$$

Realising that if we make the following substitutions instead,

$$z = \beta_0 - P; \qquad A_i = \beta_0 - \alpha_i \qquad (i = k, l, m);$$

$$B_j = \beta_j - \beta_0 \qquad (j = s, t); \qquad 0 \leq z \leq n; \qquad n = \beta_0 - \alpha_0,$$

where the indices $i = k, l, m$ and $j = s, t$ are used for those values of the α's and β's other than α_0 and β_0, we showed (Srinivasa Rao and Venkatesh 1977) that an equivalent symbolic binomial expansion can be derived:

$$\begin{aligned} W(abcd; ef) = {} & N(-1)^{\beta_1-\beta_0}\Gamma(\beta_0+2) \\ & \times[\Gamma(n+1, A_k+1, A_l+1, A_m+1, B_s+n+1, B_t+n+1)]^{-1} \\ & \times\{(B_s+n)(B_t+n) - A_kA_lA_m(\beta_0+1)^{(-1)}\}^{(n)}; \end{aligned} \tag{6}$$

where we have used the notation:

$$p^{(x)} = p!/(p-x)! \qquad \text{and} \qquad p^{(-x)} = 1/p^{(x)}$$

and regarded the form $p^{(x)}$ to represent the generalised powers of p.

Since the derivation of (6) uses the generalised power and the notation $p^{(-x)} = 1/p^{(x)}$, while Sato's derivation of (5) uses the generalised power and the notation $p^{(-x)} = (p+x)!/p!$ it follows that (6) is an exact binomial expansion at least for the power $n = 1$. It is straightforward to show, as in the case of the 3-j coefficients, that these structure zeros which arise from the polynomial part of the 6-j coefficient are indeed polynomial zeros of degree one. Therefore, we suggest that the conventional definition given by (4) may be modified to include a multiplicative factor $(1-\delta_{x,y})$ where

$$x = (\beta_s - \alpha_0)(\beta_t - \alpha_0)(\beta_0 + 1)$$

and

$$y = (\beta_0 - \alpha_k)(\beta_0 - \alpha_l)(\beta_0 - \alpha_m).$$

Since, for $n = 1$, $\beta_0 + 1 = \alpha_0 + 2$ and $(\alpha_0+1)^{(-1)}$ in Sato's notation is (α_0+2); it can be shown that this same condition can also be derived from (5).

Biedenharn and Koozekanani (1974) calculated the zeros of the 6-j coefficient using a computer program, based on the powers of primes. Their tables for the non-trivial zeros of $\{\begin{smallmatrix} a & b & c \\ d & c & f \end{smallmatrix}\}$ have been given for all arguments $(a, b, c, d; e, f)$ being $\leqslant 18.5$, and the arguments are ordered in a 'speedometric fashion'. On the other hand, we generated the polynomial zeros of degree one, using the simple arithmetic condition stated above, without actually calculating the 6-j coefficient itself. Our calculation, for all arguments $\leqslant 10$, revealed 188 zeros and these account for a large number (188 out of 216) of those listed in the tables (Koozekanani and Biedenharn 1974) as 'non-trivial' zeros.

In conclusion, we state that the polynomial zeros of degree one are in fact trivial zeros revealed by the structure of the 3-j and 6-j coefficients, when they are cast into their corresponding symbolic binomial forms. We suggest that simple multiplicative factors $(1-\delta_{x,y})$ are introduced into the definitions of these coefficients to explicitly exhibit these trivial structure zeros, which were hitherto considered as 'non-trivial' zeros along with other polynomial zeros of higher degrees.

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